A consolidated, efficient and practical model for American options with discrete dividends

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Summary

The binomial model for pricing American option is well developed with a tremendous volume of research on the topic. The original model does have several shortcomings which have been addressed by improvements and modifications.

However majority of the research has focused on independently analyzing one particular area or issue with the model, for example convergence, accuracy of greeks and handling of discrete dividends.

While each of these issues have several reasonable independent solutions, the goal of this paper is to construct one consolidated model that is efficient and accurate. The various models that fix single issues cannot be blindly “stitched” together without possible unintended affects.

Several approaches mentioned below, in particular while addressing the discrete dividend issue, rely on a large number of iterations. The definition of efficiency is a practical model, that can be used in real time trading and risk applications to calculate options value, greeks and implied volatilities fast.

In this paper, the consolidated model developed will address several issues outlined below.

One well known issue with the binomial model is that of oscillating prices and convergence only after a large number of iterations. The standard binomial model typically requires several hundred iterations to achieve a level of acceptable convergence – which is impractical when using it for valuing a large number of options in real time. Several methods have been developed to encourage faster convergence, including Jarrow and Rudd, Tian and Leisen and Reimer.

Another identified issue with the binomial model is in the calculation of the greeks- in particular the Delta and Gamma. In the standard implementation of the Cox, Ross, Rubenstein binomial tree, Hull calculates the delta and gamma based on the 2nd and 3rd nodes of a tree respectively and thus they are not contemporaneous in time with the valuation of the option. Pelsser and Vorst propose a method to resolve this.

The biggest challenge is American options with discrete dividends. When using a binomial tree to accurately incorporate the discrete dividend, the tree becomes non-recombining – which increases the number of nodes dramatically. This renders the non-recombining binomial tree useless in practice as the calculation time is too large.

The standard approach is a divided escrow approach or approximation models such as Roll-Geske-Whaley (only applicable to American calls). Such models perform reasonably well in many cases, but are quite inaccurate in the cases of multiple dividends, certain in the money options, longer dated options and dividends close to maturity etc. Several alternate approaches has been developed, including volatility adjustment along with the escrow model, such as Chriss or models such as Haug and Haug. In their colorful paper, Haug, Haug and Lewis discuss the discrete divided issue in detail and propose an alternate solutions. The Vellekoop-Nieuwenhuis model approaches the problem in a totally different way resulting in a very capable solution.

The paper will explore these issue, suggest the recommended solution for each problem and propose a consolidated generalized model than encompasses all the enhancements with necessary adjustments to make them work together for all types of options.

1.0 The Binomial model

The standard Cox-Ross-Rubenstein binomial model

$u = e^{\sigma\sqrt{T-t}}$ $\qquad d = e^{-\sigma\sqrt{T-t}}$ $\qquad$ Thus $d = \frac{1}{u}$

$$p = \frac{e^{(r-D)(T-t)} - d}{u - d}$$

Using backwards induction, the price of an American call or put can be determined through the following:

$$C_{i,j} = Max\{Su^j d^{i-j} - X, e^{-r(T-t)}[pC_{i+1,j+1} + (1-p)C_{i+1,j}]\}$$

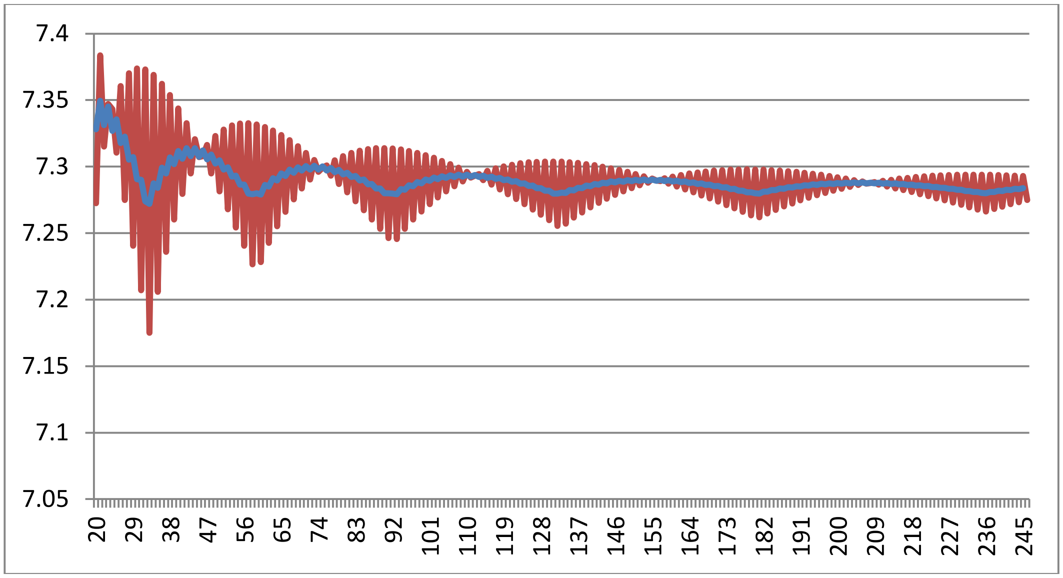
$$P_{i,j} = \max\{X - Su^j d^{i-j}, e^{-r(T-t)}[pP_{i+1,j+1} + (1-p)P_{i+1,j}]\}$$

In the above formula, D is used for both the dividend yield (continuous yield dividend) and borrowing rate.
Thus D = b + d where b is the borrowing rate and d is the dividend yield

2.0 Convergence

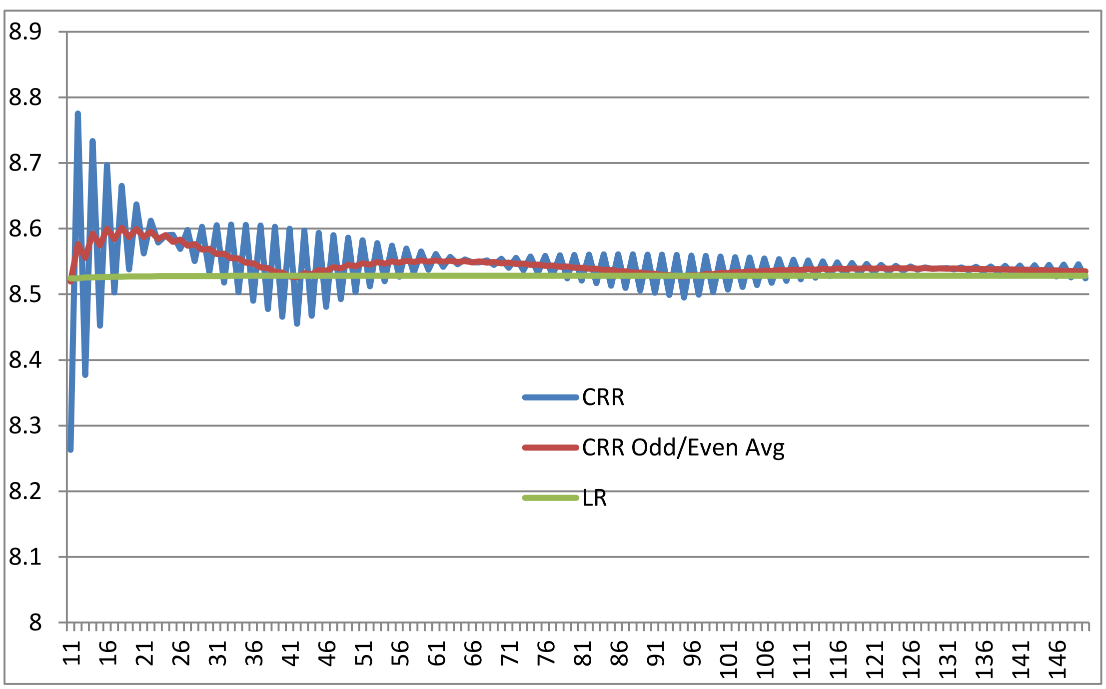
One well known issue with the standard binomial model is that of oscillating prices and convergence occurring only after a large number of iterations. This make the model less efficient as the user is forced to a large number of iterations to reduce this oscillation
One extremely simple way to solve this is the simple and elegant procedure of odd/even averaging, which simply calculates the option value for an even number of iterations and then by increasing the number of iterations by 1, and taking the average of the two results.
The issue with Odd/Even averaging is that is requires running the binomial model twice, thus while it can be used for less iterations, it is still not the most efficient. It also still suffers from some higher order harmonics and has noticeable oscillation.
Mark Joshi studied the various approaches for improving convergence.

The graph below shows the standard Binomial vs using odd/even averaging:



Our preferred method for convergence is the use of a modification to the tree parameter suggested by Leisen and Reimer. This method has exceptionally good convergence and is accurate even for a smaller number of iterations. It has become the most popular method by market makers and equity options risk/trading systems.

Below is an example of CRR vs Odd/even vs LR



The Leisen- Reimer (LR) model

The aim of the LR approach was creation of binomial trees, which beforehand, obey the sources of irregularities simply by different but only slightly modified definition of tree parameters u(n) and d(n).

The model takes the approach of starting with the terminal distribution and employing an inversion process such as Camp Paulson or Peizer Pratt to obtain p and p0 as distribution parameters components in the binomial option pricing formula. Then the tree parameters u(n) and d(n) are derived by a simple trick. The no arbitrage condition implies that

$p(n) = (r-d(n))/ (u(n)-d(n))$ still holds.

Furthermore, p0 is defined to p0 = u/ r. p. Taking these two relations as equation system which can be solved uniquely

The formulas below sum up to the model parameters. Notice, that f (z; j(n)) denotes the chosen inversion function.

$p0 = f (d1; j(n))$
$p = f (d2; j(n))$
$u = r. p0(n) /p(n)$

$d =r - p(n) . u(n) / 1 - p(n)$

The resulting binomial tree parameters diverge only very little from those of previous models, but the convergence properties with the computation of option prices changes dramatically.

One important distinction in the LR adjustments is the modification of the “up” and “down” parameters of the binomial model such that u <> 1/d. Thus the tree is no longer perfectly symmetrical, rather appears a tiny bit twisted.

3. Greeks: **Delta / gamma for tree models:**

In the standard CRR model, the Delta calculation is typically done by using 1 iteration (time slice) forward, and the Gamma will be over 2 time slices. If the number of iterations is very large and/or time to expiration is relatively short, this will make little difference.

But say you have a 100 day option with 50 iterations – the nodes used for Delta will thus span 2 days and for Gamma will span 4 days. This will lead to “jumpy” delta and gamma as they are not contemporaneous in time with each other or with the valuation of the option.

Most commercial models ignore this or use the slope method which can suffer from a similar problem. In fact the slope method can be very sensitive to the choice of slope parameters and can be “jagged” for moves in the underlying.

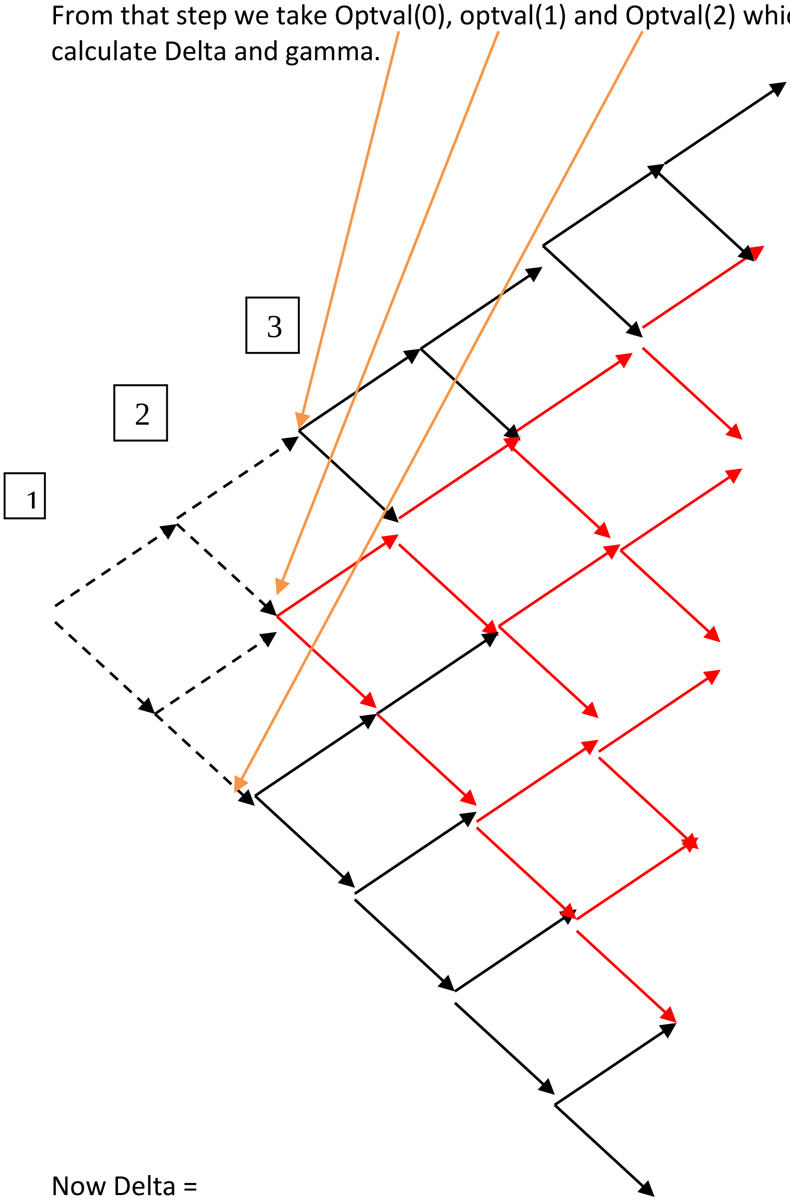
In the case of large discrete dividends the slope or simple tree node method can start producing very jumpy results.

To fix this, we use a method outline by Vorst and Ploesser in 1995 in which the tree is extended is extended backwards by 2 iterations. Thus in a perfectly symmetrical tree, the center node for the second node forward has the same spot rate as the starting node, but since there are 2 nodes around it, we can use that slice to calculate the option value, delta and gamma. This ensures that the delta and gamma are perfectly contemporaneous with the options value.

Say our original tree is in RED

The Extended tree is Black – i.e. it is extended backwards by 2 steps (extension = 2)

When we recurse backward to step J > extension, thus to step 3 only.
From that step we take Optval(0), optval(1) and Optval(2) which are the 3 nodes at that step, to calculate Delta and gamma.



Now Delta =
{At iteration 3 of extended tree}
(OptVal(2) - OptVal(0)) / ((u ^ 2) - (d ^ 2))

And Gamma =
{At iteration 3 of extended tree}

(((OptVal(2) - OptVal(1)) / ((u ^ 2) -1)) – ((OptVal(1) - OptVal(0)) / (1- (d) ^ 2)))
/ (0.5 * ((u ^ 2) – (d ^ 2)   )) / S

This method is very elegant in that it only requires extending the tree 2 steps. It resolves the problem of delta and gamma moving in time – as they are calculated at the same point in time – resulting in stable, accurate delta/gamma. It is also more efficient than the slope method as the model does not need to be called twice.

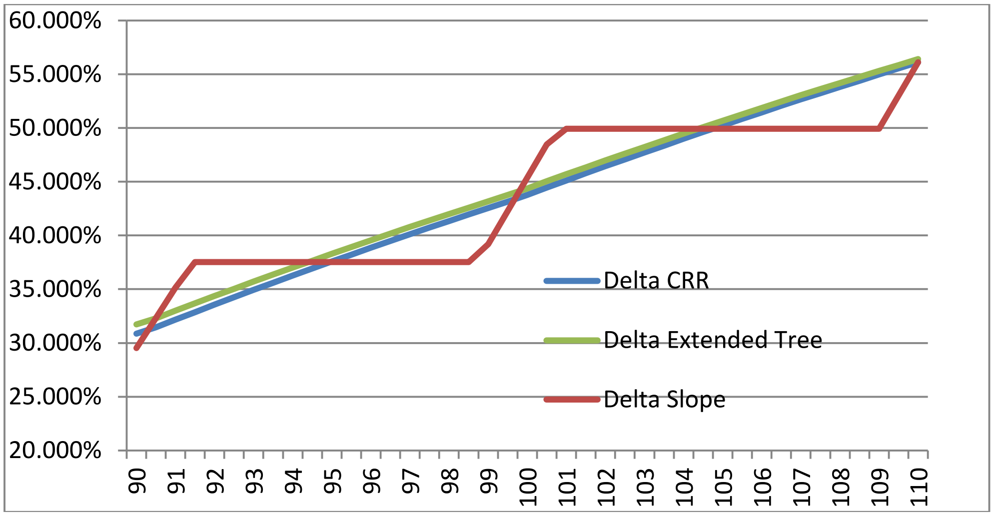


Chart: Delta from the standard CRR is slightly lower that delta from extended tree, which is intuitive because the CRR delta is actually a delta forward in time and thus has small delta decay
The slope method is very unreliable and sensitive to the actual slope used and iterations

4. Discrete dividends:

The biggest challenge is American options with discrete dividends. The most accurate model is the non-recombining binomial tree, however at the dividend step, since the tree becomes non-recombining – the number of nodes increases dramatically. This renders the non-recombining binomial tree useless in practice as the calculation time is too large. In reality the non-recombining tree has several complex issues, summarized in Appendix A.

Thus most implementations rely on either a **dividend escrow approach** (with or without volatility adjustments) or approximation models such as Roll-Geske-Whaley (only applicable to American calls). Such models perform reasonably well in most cases, but are quite inaccurate in the cases of multiple dividends, certain in the money options, longer dated options and dividends close to maturity etc.

For European Calls and puts with discrete dividends (no stock options are European, but we will discuss this for completeness sake), a simply "escrow dividend" model can be used which discounts the dividends to present value and makes an adjustment to spot.
Thus the escrow model effectively preserves the forward price by adjusting spot.

The dividend escrow model takes the present value all dividends until expiration and adjusts the spot rate and then builds the tree. On the backward recursion, it calculates the option value using dividends only till the node in question.
The issue with this approach is that it actually changes the distribution because it is changing spot. Also it does not perform well when dividends are close to expiration.

The issue with this is as follows:
Say there is one dividend at time t' with final maturity t.
Spot is S

The correct evolution of the stochastic process is based on the volatility applied to the Spot price till t' and thereafter the volatility applied to the (forward price – D) .

By adjusting the Spot level down, the stock price process is modified to be the volatility applied to the adjusted Spot price for the entire period t. This reduces the total absolute stock price variance. The difference can be small and insignificant – but in some cases it can be significant.

E.g.

a) when dividend is large, the adjustment to Spot is large, thus making the adjusted process deviate from the correct process
b) when the volatility is high- similar effect
c) when dividend Is far away in time, i.e. closer to maturity – then a majority of the tree is created using a wrong Spot price.

Several researchers, such as Chriss and Haug and Haug have proposed a small volatility adjustment for more accuracy, which makes sense, but cannot fix the issue in all cases.

Haug, Haug and Lewis that emphasize the weakness of the escrow and volatility adjustment methods in their paper

The Vellekoop-Nieuwenhuis (VN) model does not change the spot and thus the distribution is developed from the spot rate. In this approach, a normal binomial tree is first created using the current spot price. Then when the backward recursion is being performed, when you hit a dividend date, apply an adjustment to the option price at each node based on the dividend amount at that step. Then continue the recursion backwards until another dividend date it hit or until complete. Thus the recombining tree is maintained.

There are other similar approaches that use multiple distributions and even double integrals – conceptually similar – e,g. the HHL model by Haug, Haug and Lewis. However the VN model has shown to be the most versatile and accurate so far.

The VN paper simply suggests using a linear function to adjust the option price at the various nodes at the dividend step. They suggest to a linear function based on the spot and option price at the nodes above and below, knowing that the option price is floored at 0 and max at intrinsic. While their approach seems to work very well given the results in the paper, it has some practical shortcomings and issues that are not explicitly handled:

- the paper uses a large number of iterations for all their experiments. This is impractical from a performance perspective and so the challenge was to modify their methodology to perform well with fewer iterations
- When using less iterations, in cases where the dividends were soon, the number of nodes available for creating the linear function were too few – resulting in large inaccuracy.
- Their model suggest using a linear function, however the sensitivity of an option price to a change in spot is not linear but convex. In the case of large dividends and lower iterations, their method resulted in inaccurate results.
- We have seen implementations in which the author used culic spline interpolation as opposed to linear with improved results.
- Boundary conditions need to be explicitly handled in the model implementation, in particular in the case of puts (incidentally they only use Call options in their paper), in order to avoid negative option prices (and thus negatively probability.
- Similarly when performing the interpolation for the dividend date, appropriate assumptions are required for nodes at bottom of the tree

5. The Consolidated model

In the consolidated model we start with the VN model as our baseline, as it solves the larger issue of handling discrete dividends. The model performs very accurately for even long term options with many dividends as is shown in the original VN paper.

However, the model prefers to have a large number of iterations available in order to reach convergence. So the first order of business in the consolidated model was to make the VN model more efficient in terms of iterations.

The VN is essentially a standard binomial (CRR) with modified handling of the option value at the node in the tree where a discrete dividend occurs. It does not modify the volatility process to solve the discrete dividend problem (as some other models attempt to do) nor does it change the terminal distribution. The end distribution is evolved from the actual spot rate, as opposed to an adjusted spot rate as in the escrow models. Thus the starting point distribution is pure – which is the basis for applying the LR adjustments for faster convergence. Thus our consolidated model first improvement is to apply the LR adjustment within the VN model

Appendix B shows sample results of the VN model with LR adjustments with comparison to the results from the VN paper. We clearly show that similar results are obtained with fewer iterations, thus achieving our goal of a more efficient model.

Note: our model results vary slightly from the VN paper. We attribute this to possibly the following:

- The VN paper does not explicitly state their interpolation methodology
- When assigning a dividend at a given time to a specific node (discretization), there is a choice of assigning to the closest node, the prior node or later node. We do not know the VN paper assumption

Both these lead to extremely small differences that are irrelevant to this discussion

Interpolation and Boundary conditions in the VN model:
The VN model is very sensitive to interpolation of the option value at the iteration of a discrete dividend.

Assume a divided occurs at iteration = i

At this iteration there exists i+1 nodes, with :

$S(0) \ldots S(i+1)$ spot prices and $OV(0) \ldots OV(i+1)$ option values.

Depending on the implementation, the spot prices can be high to low, or low to high. For this paper we assume $S(0) > S(1) \ldots. > S(i+1)$

Assume the discrete dividend is D

Thus for each node $S(n)$, where n = 0 to i+1, we search for node k, where

$$S(k) >= S(n) - D \text{ and } S(k+1) <= S(n) - D \qquad (1)$$

i.e we search for the nodes within which the dividend adjusted spot rate falls.

Then the simple interpolation is :

$$OV(n) = OV(n) + [\ OV(k+1) - OV(k) * S(n)-D - S(k)\ ] / S(k+1) - S(k)$$

For all cases where $k <= i$

However in the case of a large dividend D or a limited number of nodes, i+1, we may not be able to find a k that satisfies (1) or k may be > i. In such a situation we have a boundary condition that needs to be explicitly dealt with.
The constraints we apply:

$S(n) - D <= 0$
For Put options, $OV(n) <=$ Strike
When k = i+1, interpolation is modified to be the following (as then k+1 is not available)

$$OV(n) = OV(n) + [\ OV(k) - OV(k-1) * S(n)-D - S(k)\ ] / S(k) - S(k-1)$$

The interpretation for the interpolation is the calculation of a linear slope (delta) for the option prices straddling the spot rates in between which the adjusted spot lies. In the case of hitting the boundary of the tree, we use the closest delta available

These adjustments ensure avoiding negative spot rates and negative probabilities. These conditions are critical to valuation of Put options as we are making downward adjustments to the spot prices due to dividends and thus at some point hit the lower boundary of the tree, at which point the put options have maximum value.
In the case of call options, they have lowest or 0 value at the lower boundary of the tree and thus the impact of the boundary conditions is minimal
The upper bounds of the tree are irrelevant as the spot prices are being adjusted downward only.

Incidentally, the VN paper only calculates Call prices and thus completely avoids the issue of the above boundary conditions.

Results: Appendix B shows results for Calls and Puts for a variety of options compared to a non-recombining tree where available and a large grid finite difference model. The results compare extremely favorably in all cases.

Delta, gamma calculations: the final enhancement was for calculating accurate delta and gamma. The method chosen is the Vorst and Ploesser backward extended tree as described in section 3 above. However the assumption is that the tree is symmetrical, i.e. u = 1/d
However as described in section 2, the LR adjustments modify u and d such that u <> 1/d and thus the tree is no longer perfectly symmetrical

Thus the extended tree method cannot be directly applied to the LR model.

The adjustment is actually quite simple. For valuation purposes, we are only concerned with iteration 3 and beyond of the extended tree. The center point node spot rate at this iteration should be the Spot rate for the options at hand.

Assume actual spot rate is S.
At iteration 3, we have S(1), S(2) and S(3) as the 3 spot rates.
We need S(2) = S
S(2) by definition is S' * u*d
Where S' is the spot rate used to build the extended tree. If the tree was symmetrical and u = 1/d, then S = S'
But since u <> 1/d, we simply set S' = S / u*d and build the extended tree starting with S'

This simple adjustment makes It possible to use the extended tree calculations for delta and gamma possible with the LR adjustments

Conclusion

The final model is an efficient model for American options with discrete dividends which combines several different techniques for solving various known issues in the binomial pricing framework. The result is one model encompassing the adjustments with additional adjustments to ensure they all work well together. It is a generalized model and can be used for calls and puts as it handles boundary conditions as well.

Appendix A

Issues in Non-Recombining tree modelling

The path to getting an “accurate” pricing model for American options with multiple discrete dividends is not easy!
Literature suggests a non-recombining tree is the most accurate, but several issues actually exist is the use of it:

1. Constraint: the main constraint is the number of iterations. Even with 200 or so iterations and 2 div, the number of nodes can explode to an unwieldy 250,000 or so – which makes the model take forever to calculate

2. Oscillations: the main issue that occurred is the oscillation of the options price when iterations are changed (the odd / even issue in a binomial tree). While a normal binomial should exhibit little oscillation when iterations are in the 200 range, with the non-recombining, each of the sub-trees have far less than 200 each – thus they will show oscillations.

3. Different sub trees with different odd/even iterations: say there are 201 total. The first div falls at say iteration 100 and the second at 150. The 3 sub-trees have iterations of 100, 50, 61 - i.e. even/even/odd. Changing this to 202 could result in the divs falling at 101 and 150 making the iterations 101, 49 and 62 i.e. odd/odd/even. The result is a complex harmonic oscillation.

4. LR adjustment: to smooth the oscillations, the LR adjustment could be applied.. Given there are many sub trees (with 201 iterations, we could have over 5000 sub trees), each with a different distribution, applying the LR adjustments would require calculating the parameters for each sub tree.

Appendix B

Results